Quaternions in Electrodynamics

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At the advent of MAXWELL's electrodynamics the quaternion notation was often used, but today this is replaced in all text books with the vector notation. If the founders of electrodynamic would have used the quaternion notation consequently with their most unique property – namely the four-dimensionality – they would have discovered relativity much before VOIGT, LORENTZ and EINSTEIN. A short description of electrodynamic with quaternions is given. As a result a new set of MAXWELL's equations is proposed, which transform in today's equations when the LORENTZ gauge is applied. In addition an application of this new quaternion notation to quantum mechanics and other disciplines is presented.

Introduction

One of the most emotional disputes in the late nineteenth-century electrodynamics was about the mathematical notation to use with electrodynamic equations [1]. [1]. The today's vector notation was not fully developed at that time and many physicist – one of them was James Clerk MAXWELL – are convinced to use the quaternion notation. The quaternion was "invented" in 1843 by Sir William Rowan HAMILTON [5]. Peter Guthrie TAIT[11] was the most outstanding promoter of quaternions. On the other side Oliver HEAVISIDE [6] and Josiah Willard GIBBS[12] both decided independently that they could use a part of the quaternion system better than the entire system, why they proceeded further with that, what is today called the vector notation. Generally the vector notation used in pre-EINSTEIN electrodynamics uses three-dimensional vectors. The quaternion on the other hand is a four-dimensional number. To make the quaternion usable for the three-dimensional electrodynamic of MAXWELL, HAMILTON and TAIT indicated the scalar part by prefixing an 'S' to the quaternion and the vector part by prefixing a 'V'. This notation was also used by MAXWELL in his Treatise[8] , where he published twenty quaternion equation with this notation. But with applying this prefixes the whole benefit of quaternions is not used. Therefore MAXWELL has never done calculations with quaternions but only presented the final equations in a quaternion form. It was then merely a calculation with vectors and scalars as today practiced.

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HAMILTON's Quaternions

A general quaternion has both a scalar (real) and a vector (imaginary) part. In the example below 'a' is the scalar part and 'ib + jc + kd' is the vector part.

$$\mathbb{Q} = a + bi + cj + dk \tag{1}$$

Where a, b, c, and d are real numbers and i, j, k are so-called HAMILTON'ian unit vectors with the magnitude of √-1. They satisfy the equations

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

and

$$ij = k \quad jk = i \quad ki = j$$
$$ij = -ji \quad jk = -kj \quad ki = -ik$$

A nice explanation about the rotation capabilities of the HAMILTON'ian units in a three-dimensional ARGAND diagram was published by GOUGH [3].

A quaternion is a hypercomplex number. The quaternion radii (or magnitude) in four-dimensional space is defined similar as for ordinary complex numbers as:

$$|\mathbb{Q}| \equiv \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \tag{3}$$

as also shown by Walker[12]. By introducing a conjugate quaternion number

$$\mathbb{Q}^* = a - bi - cj - dk \tag{4}$$

the quaternion magnitude is also

$$|\mathbb{Q}| \equiv \sqrt{\mathbb{Q}\mathbb{Q}^*} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \ . \tag{5}$$

The four-dimensional quaternion is very suitable to represent an event in four-dimensional vector space†:

$$^4\mathbf{X} = \left(x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\right) \quad \Leftrightarrow \quad \mathbb{X} \equiv x_0 + x_1 i + x_2 j + x_3 k \ . \tag{6}$$

Expansion of Quaternions with Imaginary Numbers

An expansion of quaternions to eight-dimensional numbers can be achieved when the real variables a, b, c and d now are replaced with imaginary numbers. A complex quaternion is then:

$$\mathsf{Q} = (a + \mathrm{i}A) + (b + \mathrm{i}B)i + (c + \mathrm{i}C)j + (d + \mathrm{i}D)k \tag{7}$$

Because the HAMILTON'ian unit vectors i, j, k are still valid beside the imaginary (GAUSS'ian) unit i this new introduced quaternion is different from the known octionions (known form LIE algebra). A complex quaternion is a number which represents two intersecting four-dimensional spaces. The complex quaternion Q can be splitted into two sub entities:

$$\begin{aligned}\mathsf{Q} &= \left(X_0 + \mathrm{i}x_0\right) + \left(x_1 + \mathrm{i}X_1\right)i + \left(x_2 + \mathrm{i}X_2\right)j + \left(x_3 + \mathrm{i}X_3\right)k \\ &= \left(\mathrm{i}x_0 + x_1 i + x_2 j + x_3 k\right) + \left(X_0 + \mathrm{i}X_1 i + \mathrm{i}X_2 j + \mathrm{i}X_3 k\right) = \mathbb{Q} + \overline{\mathbb{Q}}\end{aligned}$$

The first term (upper line in above equation) is very suitable for a compact notation of electrodynamics if the second term (lower line) is always set to zero as some investigations have shown. With this the complex quaternion reduces again to a four-dimensional number with the specialty, that no real number exists anymore. It will be shown, that this complex quaternion is also useful for other applications.

† The unit vectors in three-dimensional space are printed in bold letters with **i**, **j**, **k**, the HAMILTON'ian units i, j, k in italic letters and the imaginary unit i in normal letters without serifs.

Definitions

The following definitions applies always to the complex quaternion Q.

Definition 1: A complex quaternion is splitted into two parts, whereas the second part is always set to zero according to:

$$\begin{aligned}\mathsf{Q} &= \left(X_0 + \mathrm{i}x_0\right) + \left(x_1 + \mathrm{i}X_1\right)i + \left(x_2 + \mathrm{i}X_2\right)j + \left(x_3 + \mathrm{i}X_3\right)k \\ &= \left(\mathrm{i}x_0 + x_1 i + x_2 j + x_3 k\right) + \left(X_0 + \mathrm{i}X_1 i + \mathrm{i}X_2 j + \mathrm{i}X_3 k\right) \\ &\equiv \left(\mathrm{i}x_0 + x_1 i + x_2 j + x_3 k\right) = \mathbb{Q}\end{aligned} \tag{8}$$

Definition 2: The quaternion Nabla operator is:

$$\blacktriangledown \equiv \frac{\partial}{\partial \mathbb{X}} = -\frac{\mathrm{i}}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k \tag{9}$$

Definition 3: The quaternion LAPLACE (or D'ALEMBERT) operator is:

$$\blacksquare \equiv -\left|\blacktriangledown\right|^2 = -\blacktriangledown\blacktriangledown^* = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla \bullet \nabla \tag{10}$$

Definition 4: The total derivative of time is:

$$\frac{d}{dt} = -\mathbb{V}\blacktriangledown \tag{11}$$

The scalar part is analogue to the well-known equation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\nabla$$

Definition 5: The amount (radii) of a quaternion is:

$$\left|\mathsf{Q}\right| \equiv \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \tag{12}$$

Quaternions in four-dimensional Space

In 1912 the first attempt to use quaternions for relativity was published by SILVERSTEIN[10]. He used two quaternion operator to formulate relativity. But from the presentation above about the flat four-dimensional space an analogue representation of the curved four-dimensional MINKOWSKI space can be given with one single quaternion with:

$$^4\mathbf{X} = \left(\mathrm{i}ct + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\right) \quad \Leftrightarrow \quad \mathbb{X} \equiv \mathrm{i}ct + x_1 i + x_2 j + x_3 k \ . \tag{13}$$

The magnitude (radii) according to (3) leads to:

$$X = \left|\mathbb{X}\right| = \sqrt{x_1x_1 + x_2x_2 + x_3x_3 - c^2t^2} \tag{14}$$

This magnitude is different to the definition 5, but is according to the special theory of relativity invariant to transformations between inertial systems. This applies also for the differential dX. A division with $\mathrm{i}c$ results in another invariant:

$$\frac{1}{\mathrm{i}c}dX = \sqrt{dt^2 - \frac{1}{c^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right)} = dt\sqrt{1 - \frac{v^2}{c^2}} = d\tau \tag{15}$$

This is the time dilatation known form relativity. For the differential of an event we can further write:

$$d\mathbb{X} = \mathrm{i}cdt + i\,dx_1 + j\,dx_2 + k\,dx_3 \ . \tag{16}$$

so that we get the known four-dimensional velocity $\mathbb{U}$:

$$\mathbb{U} = \frac{d\mathbb{X}}{d\tau} = \frac{\mathrm{i}c}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{i\,v_1 + j\,v_2 + k\,v_3}{\sqrt{1-\frac{v^2}{c^2}}} \tag{17}$$

The derivation of the time dilatation follows from the magnitude of an event vector, whereas the derivation of the four-dimensional velocity is applied to the vector itself. This is not consistent as Yong-Gwan YI[15] already has found. The concept of time dilatation and of the four-dimensional velocity (17) above do exclude each other.

Without the concept of curved space the four-dimensional velocity can also be formulated in an other way:

$$\mathbb{V} = \frac{d\mathbb{X}}{dt} = \mathrm{i}c + i\,v_1 + j\,v_2 + k\,v_3 \tag{18}$$

This effective (absolute) velocity of a body in flat space should now be used in the further explanations.

Quaternions in Electrodynamics

Analogue to the velocity we define the quaternion potentials with:

$$\mathbb{A} \equiv \mathrm{i}\frac{\varphi}{c} + i\,A_1 + j\,A_2 + k\,A_3 \tag{19}$$

The quaternion current is then:

$$\mathbb{J} \equiv \mathrm{i}c\rho + i\,J_1 + j\,J_2 + k\,J_3 \tag{20}$$

So we get with the potential:

$$\blacktriangledown\mathbb{A} = -\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) + \left(\frac{\mathrm{i}}{c}\frac{\partial A_1}{\partial t} + \frac{\mathrm{i}}{c}\frac{\partial\varphi}{\partial x_1} + \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)i$$
$$+\left(\frac{\mathrm{i}}{c}\frac{\partial A_2}{\partial t} + \frac{\mathrm{i}}{c}\frac{\partial\varphi}{\partial x_2} + \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)j + \left(\frac{\mathrm{i}}{c}\frac{\partial A_3}{\partial t} + \frac{\mathrm{i}}{c}\frac{\partial\varphi}{\partial x_3} + \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)k$$

Then with the substitutions

$$-\mathbf{E} = \nabla\varphi + \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A} \tag{21)a, b}$$

we get the compact formula

$$\blacktriangledown\mathbb{A} = -\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\bullet\mathbf{A}\right) + \left(\frac{-\mathrm{i}}{c}E_1 + B_1\right)i + \left(\frac{-\mathrm{i}}{c}E_2 + B_2\right)j + \left(\frac{-\mathrm{i}}{c}E_3 + B_3\right)k \tag{22}$$

According to definition 1 the real scalar term is zero together with the imaginary vector part. From this results the known LORENTZ condition (gauge):

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\bullet\mathbf{A} = 0 \tag{23}$$

$$\text{with} \quad \mathbf{B} = \nabla\times\mathbf{A} \quad \text{and} \quad \mathbf{E} = 0$$

With definition 1 the LORENTZ condition is strictly valid for $\mathbf{E} = 0$, but otherwise it can be freely chosen. According to the definition of the potentials (21) it is possible to choose an arbitrary definition because of the divergence of the vector potential **A**, but for $\mathbf{E} = 0$ the LORENTZ condition is ultimate. Then for the current density we can derive:

$$\blacktriangledown\mathbb{J}=-\left(\frac{\partial\rho}{\partial t}+\frac{\partial J_1}{\partial x_1}+\frac{\partial J_2}{\partial x_2}+\frac{\partial J_3}{\partial x_3}\right)+\left(\frac{i}{c}\frac{\partial J_1}{\partial t}+ic\frac{\partial\rho}{\partial x_1}+\frac{\partial J_3}{\partial x_2}-\frac{\partial J_2}{\partial x_3}\right)i$$
$$+\left(\frac{i}{c}\frac{\partial J_2}{\partial t}+ic\frac{\partial\rho}{\partial x_2}+\frac{\partial J_1}{\partial x_3}-\frac{\partial J_3}{\partial x_1}\right)j+\left(\frac{i}{c}\frac{\partial J_3}{\partial t}+ic\frac{\partial\rho}{\partial x_3}+\frac{\partial J_2}{\partial x_1}-\frac{\partial J_1}{\partial x_2}\right)k \tag{24}$$

Here we can also introduce the substitutions:

$$-\mathbf{G}=c^2\nabla\rho+\frac{\partial\mathbf{J}}{\partial t}, \qquad \mathbf{C}=\nabla\times\mathbf{J} \qquad \text{(25)a, b}$$

whereas follows the compact formula

$$\blacktriangledown\mathbb{J}=-\left(\frac{\partial\rho}{\partial t}+\nabla\bullet\mathbf{J}\right)+\left(\frac{-i}{c}G_1+C_1\right)i+\left(\frac{-i}{c}G_2+C_2\right)j+\left(\frac{-i}{c}G_3+C_3\right)k \tag{26}$$

Again according to definition 1 the real scalar and the imaginary vector part is set to zero. From this follows the known continuity equation (conservation of electrical charge):

$$\frac{\partial\rho}{\partial t}+\nabla\bullet\mathbf{J}=0 \tag{27}$$

$$\text{with}\quad \mathbf{C}=\nabla\times\mathbf{J}\quad\text{and}\quad\mathbf{G}=0$$

New for this formulation is, that the continuity condition – similar to the LORENTZ condition – is only strictly valid for $\mathbf{G}=0$, but elsewhere it can be chosen freely because of $\mathbf{C}=\nabla\times\mathbf{J}$.

The LORENTZ Force

If we define the quaternion momentum as

$$\mathbb{P}\equiv\frac{i}{c}W+i\,p_1+j\,p_2+k\,p_3 \tag{28}$$

then we get for the quaternion momentum of an electrical charge q in a potential field:

$$\mathbb{P}_q=-q\mathbb{A} \tag{29}$$

From this follows immediately the total energy W_q of the charge q

$$W_q=-q\varphi \tag{30}$$

as well as the three-dimensional momentum as

$$\mathbf{p}_q=-q\mathbf{A} \tag{31}$$

The, if we define the four-vector force with

$$\mathbb{F}\equiv\frac{i}{c}P+i\,F_1+j\,F_2+k\,F_3 \tag{32}$$

then we get for the force of the potential field on the free charge q:

$$\mathbb{F}_q=q\frac{d\mathbb{A}}{dt}=q\left(\mathbb{V}\blacktriangledown\right)\mathbb{A} \tag{33}$$

From this we get for the vector part (writing the prefix V.):

$$\mathbf{F}_q=V.\mathbb{P}_q=q\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)-\mathbf{v}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t}+\nabla\bullet\mathbf{A}\right)+i\left(c\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}\right)\right] \tag{34}$$

According to definition 1 the first and second term corresponds to a real physical force. The first term describes the known LORENTZ force on a free charge.

$$\mathbf{F}_q=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) \tag{35}$$

The second term corresponds the LORENTZ condition (23) and is not necessarily zero. The third term in (34) is according to definition 1 equal to zero. From this we get the know relation:

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E} \tag{36}$$

The scalar part of (33) is

$$P_q = S.\mathbb{P}_q = q\left[-\mathbf{v} \bullet \mathbf{B} + \frac{i}{c}\mathbf{v} \bullet \mathbf{E} - ic\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) \right] \tag{37}$$

Again with definition 1 the first and second term is physically existent, whereas we find for the third term:

$$\mathbf{v} \bullet \mathbf{B} = 0 \tag{38}$$

If the path of a free moving charge follows exactly the field lines of **B** there will be no force exerted on the charge. The second term depends again on the LORENTZ condition (23). The first term represents the energy intake or release (power) $P_q = dW_q/dt$ of the charge moving in an electric field:

$$P_q = \frac{dW_q}{dt} = q\mathbf{v} \bullet \mathbf{E} \tag{39}$$

The Field Transformation

Equation (36) can be inserted in (35) to write the LORENTZ equation (35) without the magnetic induction field **B**:

$$\mathbf{F}_q = q\left[\mathbf{E} + \frac{\mathbf{v}}{c^2} \times (\mathbf{v} \times \mathbf{E}) \right] \tag{40}$$

If the power $P_q = 0$ then it is $\mathbf{v} \cdot \mathbf{E} = 0$ and from (40) follows:

$$\mathbf{F}_q = q\left(1 - \frac{v^2}{c^2} \right)\mathbf{E} \tag{41}$$

This formula has also been published by MEYL[9], but it is, as shown above, only valid for the condition $P_q = 0$ (uniform motion).

MAXWELL's Equations

If the quaternion LAPLACE operator is applied to the potentials, we get with (21) and some algebra for the scalar part (using the prefix S.):

$$S.\blacksquare\mathbb{A} = -\nabla \bullet \mathbf{B} + \frac{i}{c}\left[\nabla \bullet \mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) \right] \tag{42}$$

and for the vector part

$$V.\blacksquare\mathbb{A} = \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) - \frac{i}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right) \tag{43}$$

Similar equations has been presented by HONIG[7]. From the imaginary part we get with definition 1 directly AMPERE's law:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{44}$$

Now we chose

$$\blacksquare \mathbb{A} = \mu \mathbb{J} \tag{45}$$

then with the imaginary scalar part follows with (42) the expanded COULOMB law

$$\nabla \bullet \mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) = \frac{\rho}{\varepsilon} \tag{46}$$

When forcing the Lorentz condition (23) this transforms into GAUSS's law:

$$\varepsilon \nabla \bullet \mathbf{E} = \nabla \bullet \mathbf{D} = \rho \tag{47}$$

Doing the same with equation (43) delivers the expanded FARADAY law

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) = \mu \mathbf{J} \tag{48}$$

what again transforms into a known MAXWELL equation when forcing the Lorentz condition (23):

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J} \tag{49}$$

And finally by setting the real scalar part to zero we get

$$\nabla \bullet \mathbf{B} = 0 \tag{50}$$

The expanded MAXWELL equations without forcing the Lorentz condition are (valid for linear media):

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J} - \nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) \tag{48}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{44}$$

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon} - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \bullet \mathbf{A} \right) \tag{46}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{50}$$

With the classical and mostly used LORENTZ condition the expanded MAXWELL equations presented above transforms into the ordinary known set of MAXWELL's equations.

Now we take a short look at the quaternion current density. With the Quaternion velocity(18) in flat space this can be expanded to:

$$\mathbb{J} \equiv \rho \mathbb{V} = \mathrm{i}c\rho + i\ J_1 + j\ J_2 + k\ J_3 \tag{51}$$

This means, a charge density ρ is moving through four dimensional space. This corresponds exactly to the concept of an electric current.

Kinematics – analogue to Electrodynamics

The momentum of a velocity field $\mathbb{U}$ on a mass m is in quaternion notation

$$\mathbb{P}_m = m\,\mathbb{U} \tag{52}$$

The analogue writing with (29) is clear to see. It is possible to write the Equations of Electrodynamics similar to those of fluid mechanics because the electric potential field follows the same momentum laws as the velocity fields with masses. The electric four potential field corresponds to the velocity field and the charge q corresponds to the mass m.

In the definition of the charge momentum (29) an external field is the cause whereas normally in the definition for the mass momentum the velocity of the mass (relative to an observer) is taken. Therefore the momentum laws for mass and charge are not defined exactly in the same way. In this section this is now changed. The four-dimensional velocity of a mass m is then transformed into an external velocity filed:

$$\mathbb{U} \equiv \mathrm{i}\frac{c^2}{c} + i\,U_1 + j\,U_2 + k\,U_3 = \mathrm{i}c + i\,U_1 + j\,U_2 + k\,U_3 = \mathrm{i}c + \mathbf{U} \tag{53}$$

From this we get wit the definition of the momentum (28) the total energy of a mass m

$$W_m = mc^2 \tag{54}$$

as well as the three-dimensional momentum

$$\mathbf{p}_m = m\mathbf{U} \tag{55}$$

In four-dimensional space each body moves along a world line, as also defined in MINKOWSKI space. The definition of an absolute velocity **U** doesn't make sense in physics, but merely the definition of a relative velocity **u** between two bodies. This relative velocity follows form the difference of the four-dimensional velocity of two bodies:

$$\mathbb{U} = \mathbb{V} - \mathbb{W} = i\left(v_1 - w_1\right) + j\left(v_2 - w_2\right) + k\left(v_3 - w_3\right) = i u_1 + j u_2 + k u_3 = \mathbf{u} \tag{56}$$

It is only possible to measure the momentum with a relative velocity **u**, that means, at least two bodies are necessary to determine momentum. Therefore the momentum is also defined as:

$$\mathbf{p}_m = m\mathbf{u} \tag{57}$$

Interestingly the total energy of a mass is not derived from a relative movement between two masses but it is defined for a single mass. Therefore (54) is not valid anymore.

For the forces between masses a similar set of formulas can be written as used for electrodynamics, which describes forces between charges. The equivalent ansatz to (21) is:

$$\mathbf{G} = \frac{\mathbf{F}_m}{m} = -\nabla\phi - \frac{\partial \mathbf{U}}{\partial t} \qquad \text{and} \qquad \mathbf{T} = \nabla \times \mathbf{U} \qquad \text{(58)a,b}$$

with

$\mathbf{F}_m$:	Force on a mass m	[N] = [kg m / s^2]
G:	Kinemassic force field	[m / s^2]
U:	Three-dimensional velocity of mass m	[m / s]
ϕ:	Gravity potential	[m^2 / s^2]
T:	Rotary or oscillation field	[1 / s]

Then the kinemassic force field can be decomposed in two known parts:

$$\mathbf{G}_G = -\nabla\phi \qquad \text{and} \qquad \mathbf{G}_T = -\frac{\partial \mathbf{U}}{\partial t} \tag{59)a,b}$$

with

$\mathbf{G}_G$: Gravity field $[m/s^2]$

$\mathbf{G}_T$: Inertia field $[m/s^2]$

Then with (32) it follows for the force on a mass m with self velocity $\mathbb{V}$:

$$\mathbb{F}_m = m\frac{d\mathbb{U}}{dt} = -m\left(\mathbb{V}\blacktriangledown\right)\mathbb{U} \tag{60}$$

From this we get for the vector part:

$$\mathbf{F}_m = \mathrm{V}.\mathbb{F}_m = m\left[\left(\mathbf{G}+\mathbf{V}\times\mathbf{T}\right)-\mathbf{V}\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\nabla\bullet\mathbf{U}\right)+i\left(c\mathbf{T}-\frac{\mathbf{V}}{c}\times\mathbf{G}\right)\right] \tag{61}$$

The first term corresponds to a gravitational LORENTZ force

$$\mathbf{F}_m = m\left(\mathbf{G}+\mathbf{V}\times\mathbf{T}\right) \tag{62}$$

The second term is analogue to the LORENTZ condition and the third term shows the relation:

$$\mathbf{T} = \frac{\mathbf{V}}{c^2}\times\mathbf{G} \tag{63}$$

The scalar part of (60) is:

$$P_m = \mathrm{S}.\mathbb{F}_m = m\left[\frac{i}{c}\mathbf{v}\bullet\mathbf{G} - ic\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\nabla\bullet\mathbf{S}\right)+\mathbf{v}\bullet\mathbf{T}\right] \tag{64}$$

The first term corresponds to energy consumption or release (power) $P_m = dW/dt$ of a mass, which is moving in a gravitational field:

$$P_m = \frac{dW}{dt} = m\mathbf{v}\bullet\mathbf{G} \tag{65}$$

As with the electric force also the gravitation force can be summarized when (63) is inserted in (62)

$$\mathbf{F}_m = m\left[\mathbf{G}+\frac{\mathbf{v}}{c^2}\times\left(\mathbf{v}\times\mathbf{G}\right)\right] \tag{66}$$

If the power $P_m = 0$ then is $\mathbf{v}\cdot\mathbf{G} = 0$, and from (66) follows

$$\mathbf{F}_m = m\left(1-\frac{v^2}{c^2}\right)\mathbf{G} \tag{67}$$

The forces on a mass m can be formulated analogue to electrodynamics, as shown above. But if one wished to describe forces between masses as effects of forces between charges only, it is not allowed to introduce the new potentials ϕ and **U**, which are independent of φ and **A**. It should be shown in an other paper[12] that this could probably be possible.

Kinematics – according to Newton

In opposite to the description above in NEWTON's mechanics the velocity or an acceleration of a mass is always considered relative to an other mass. From there we have the basic equation:

$$\mathbf{F}_N = m\mathbf{a} \tag{68}$$

or in quaternion notation:

$$\mathbb{F}_N = m\frac{d\mathbb{V}}{dt} = -m\left(\mathbb{V}\blacktriangledown\right)\mathbb{V} \tag{69}$$

Thus we get immediately for $\mathbb{P} = \mathrm{i}W/c + ip_1 + jp_2 + kp_3$ and $\mathbb{V} = \mathrm{i}c + iv_1 + jv_2 + kv_3$:

$$W = mc^2 \qquad \text{and} \qquad \mathbf{p} = m\mathbf{v} \tag{70}$$

Before we take a closer look to (69) the quaternion Nabla operator is now applied to the velocity, as previous done:

$$\blacktriangledown\mathbb{V} = -\nabla\bullet\mathbf{v} + \nabla\times\mathbf{v} + \mathrm{i}\frac{\partial\mathbf{v}}{\partial t} \tag{71}$$

then we get with definition 1 for the scalar part the known continuity equation

$$\nabla\bullet\mathbf{v} = 0 \tag{72}$$

which is only strictly valid for $\partial\mathbf{v}/\partial t = 0$ (imaginary vector part equal zero). Then the vector part of (69) gilt nun:

$$F_N = V.\mathbb{F}_N = m\left\{\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\left(\nabla\bullet\mathbf{v}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) - \mathrm{i}\left(c\nabla\times\mathbf{v} + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{v}}{\partial t}\right)\right\} \tag{73}$$

Again the real part represents a physical real force. This force corresponds to the inertia force of an accelerated mass, which points in opposite direction than the acceleration. The inertia force (real vector part) does correspond with the known formula of fluid mechanics of the acceleration field:

$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) \ . \tag{74}$$

Then by setting the imaginary term equal zero we get a new relation:

$$\nabla\times\mathbf{v} = \frac{\mathbf{v}}{c^2}\times\frac{\partial\mathbf{v}}{\partial t} \tag{75}$$

Again the strong analogy to the equation (36) of electrodynamics can be seen. Now, the scalar part of (69) is:

$$P_N = S.\mathbb{F}_N = m\left\{\mathbf{v}\bullet\left(\nabla\times\mathbf{v}\right) + \mathrm{i}\left(c\nabla\bullet\mathbf{v} + \frac{\mathbf{v}}{c}\bullet\frac{\partial\mathbf{v}}{\partial t}\right)\right\} \tag{76}$$

With definition 1 we can see for the real part, that $\nabla\times\mathbf{v}$ must always be perpendicular to $\mathbf{v}$ to guarantee that this term will always be zero.

Quantum mechanics

Relativistic wave equation

IN 1937 CONWAY[2] has shown a possible notation for the relativistic wave equation, if the Hamilton'ian units are used as pre- and post factors in his formulas. In this section we derive the relativistic wave equation with the herein described quaternion notation. We start with the momentum law:

$$\mathbb{P} = m\mathbb{V} \tag{77}$$

and wit the definition of the total energy of a mass m

$$E \equiv c\left|\mathbb{P}\right| \tag{78}$$

so that with that and with definition 5 the EINSTEIN formula can be derived for $\mu = 1..3$:

$$E^2 = m^2c^4 + c^2\sum_{\mu} p_{\mu}^2 \tag{79}$$

Until this point we have used a scalar for the energy, which for example takes a constant value for a resting body. But quantum physics (i.e. experiments) have shown, that this is not quite correct but that energy is merely on oscillating phenomenon and therefore must satisfy a wave equation. Therefore equation (79) can be seen as a ‚static average' equation of a collection of many „energy oscillators". But for a single particle the oscillatory behavior of its energy can clearly be observed.

Equation (79) is already in a quadratic form as this is the case also for the differential operators of a wave equation. To find the wave equation behind the energy equation, we use the established substitutions for the differentials as known in quantum mechanics:

$$E \rightarrow -\frac{\hbar}{ic}\frac{\partial}{\partial t} \quad \text{and} \quad p_{\mu} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_{\mu}} \; , \tag{80}$$

where $\hbar$ is PLANCK's constant divided by 2π. Now this differentials must be applied to a new unknown function. This is nothing else than the (dimensionless) wave function Υ. This wave function is again a quaternion:

$$\Upsilon = i\psi_0 + \psi_1 i + \psi_2 j + \psi_3 k \tag{81}$$

Then we get the Dirac relativistic wave equation by inserting (81) in (79):

$$\frac{1}{c^2}\frac{\partial \Upsilon}{\partial t^2} - \Delta\Upsilon + \frac{m^2c^2}{\hbar^2}\Upsilon = 0 \tag{82}$$

Particle without external potential fields

DIRAC has solved the energy equation (79)

$$E = \pm c\sqrt{m^2c^2 + \sum_{\mu} p_{\mu}^2} \tag{83}$$

by taking the square root with the introduction of 4x4 matrixes. On the other side equation (78) offers an other possibility without taking a square root, when directly taking the quaternion momentum. But the sign of the quaternion momentum should not change. This can be achieved with a modification of equation (78):

$$\mathbb{E} \equiv \pm c\mathbb{P} \qquad \text{then it is} \quad E = \left|\mathbb{E}\right| = \pm c\left|\mathbb{P}\right| \tag{84}$$

The two possible sign for the energy state in (83) and (84) has motivated DIRAC to postulate the existence of anti-particles – especially the positron. By inserting of the substitutions (80) into (84) an other DIRAC equation ca be obtained:

$$\hbar\left(\frac{\partial Y}{\partial x_1}i+\frac{\partial Y}{\partial x_2}j+\frac{\partial Y}{\partial x_3}k+\frac{1}{c}\frac{\partial Y}{\partial t}\right)-mcY=0\ , \tag{85}$$

On a first glance this equation differs form the original DIRAC equation because no matrixes are used. But the HAMILTON'ian units can also be written as matrixes (see appendix A). The by multiplication of (85) with this HAMILTON'ian matrixes it follows the equation system:

$$\begin{aligned}
&\frac{\hbar}{c}\frac{\partial\psi_0}{\partial t}-mc\psi_0-\frac{\hbar}{i}\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}+\frac{\partial\psi_3}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_1}{\partial t}-mc\psi_1+\hbar\left(i\frac{\partial\psi_0}{\partial x_1}-\frac{\partial\psi_3}{\partial x_2}+\frac{\partial\psi_2}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_2}{\partial t}-mc\psi_2+\hbar\left(\frac{\partial\psi_3}{\partial x_1}+i\frac{\partial\psi_0}{\partial x_2}-\frac{\partial\psi_1}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_3}{\partial t}-mc\psi_3-\hbar\left(\frac{\partial\psi_2}{\partial x_1}-\frac{\partial\psi_1}{\partial x_2}-i\frac{\partial\psi_0}{\partial x_3}\right)=0
\end{aligned}\ , \tag{86}$$

This system contains four equations for a particle without external fields as originally proposed by DIRAC. Now, in an analogue way the equation for a particle within external fields can be described.

Particle within external potential field

The momentum on a charged particle changes with an external potential field. If the momentum of an external field on a charged particle q is defined as

$$\mathbb{P}_q=-q\mathbb{A} \tag{87}$$

then it follows for its energy

$$E_q\equiv c\left|\mathbb{P}_q\right|=-cq\left|\mathbb{A}\right| \tag{88}$$

and

$$\mathbb{E}_q=\mp cq\mathbb{A} \tag{89}$$

So the total energy is now

$$\mathbb{E}=c\left(\pm\mathbb{P}\mp q\mathbb{A}\right) \tag{90}$$

Again the extended DIRAC equation follows with the substitution of energy and momentum to:

$$\hbar\left\{\left(\frac{\partial}{\partial x_1}-qA_1\right)i+\left(\frac{\partial}{\partial x_2}-qA_2\right)j+\left(\frac{\partial}{\partial x_3}-qA_3\right)k+\frac{1}{c}\frac{\partial}{\partial t}\right\}Y-\left(mc+q\varphi\right)Y=0 \tag{91}$$

Again quaternions can be used instead of matrixes.

Summary

With the introduction of the complex quaternion according to definitions 1 to 4 all basic equations of electrodynamics can be formulated in a very compact way. Furthermore there result some new suggestions about an extension of MAXWELL's equations, which are dependent on the use of the LORENTZ condition. Of course this result can not cover that there are not satisfactory explanations available at the moment, why exactly this type of a four-dimensional quaternion notation can be applied to electrodynamics successfully.

Beneath electrodynamics the quaternion is also very suitable for application in other disciplines in physics, as for example in quantum mechanics or in kinematics.

The used structure of the complex quaternion is interesting because it does not contain one single reel number anymore. There is more work necessary to bring more light into the geometrical meaning of this notation and of definition 1.

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Appendix A

According to Arthur CAYLEY it is possible to represent complex numbers as a matrix:

$$a + ib = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with} \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{92}$$

Example:

$$i^2 = ii = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \tag{93}$$

The HAMILTON'ian units of a quaternion build together with the numbers 1 and –1 a non ABEL'ian group of eight order. Its first four positive elements are:

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{94}$$

If we now replace the imaginary unit i with the corresponding matrix (92), then it follows:

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{95}$$

The square of this matrixes always returns the value –1 as requested by the definition of the HAMILTON'ian unit vectors (2).

